The Lie Algebraic Structure of Differential Operators Admitting Invariant Spaces of Polynomials

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Abstract

We prove that the scalar and 2×2 matrix differential operators which preserve the simplest scalar and vector-valued polynomial modules in two variables have a fundamental Lie algebraic structure. Our approach is based on a general graphical method which does not require the modules to be irreducible under the action of the corresponding Lie (super)algebra. This method can be generalized to modules of polynomials in an arbitrary number of variables. We give generic examples of partially solvable differential operators which are not Lie algebraic. We show that certain vector-valued modules give rise to new realizations of finite-dimensional Lie superalgebras by first-order differential operators.

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1 Introduction

There has been a significant amount of interest recently in the study of Schrödinger operators which preserve explicit finite-dimensional subspaces $\mathcal{N}$ of the underlying space of smooth functions, [25], [9], [11]. A Schrödinger operator, or more generally any differential operator T, which enjoys this property is said to be *partially solvable*. Indeed, if $\mathcal{N}$ is n-dimensional and the restriction of T to $\mathcal{N}$ is a self-adjoint operator with respect to a suitable inner product, one can then determine n eigenvalues and eigenfunctions of T algebraically, counting multiplicities. Notable examples of partially solvable Schrödinger operators in one dimension include some new families of anharmonic oscillator potentials, as well as the familiar Morse, Pöschl-Teller and Mathieu potentials, which have been thoroughly analyzed from this algebraic point of view, [21], [19], [12], [7]. It has been observed that most of the known examples of partially solvable Schrödinger operators can be written as elements of the universal enveloping algebra of a finite-dimensional Lie algebra of first-order differential operators admitting a finite-dimensional subspace of the underlying Hilbert space as a module of smooth functions. The differential operators satisfying this property are called *quasi-exactly solvable*, or more generally *Lie algebraic* if no assumption is made on the existence of such a finite-dimensional module. One can thus construct many examples of partially solvable Schrödinger operators by first classifying the Lie algebras of first-order differential operators admitting finite-dimensional modules of functions, and then looking for elements in the universal enveloping algebra which can be put in the Schrödinger form after conjugation by a suitable non-vanishing multiplication operator and a local change of the independent variable. This conjugation is referred to usually as a gauge transformation. Note that the gauge transformation will generally *not* be unitary, so that square integrability need not hold at the outset, but only for the eigenfunctions of the resulting Schrödinger operator obtained after conjugation. This program has proved to be remarkably successful in that it has enabled one to construct many new examples of quasi-exactly solvable potentials in higher-dimensions, [20], [9], as well as matrix potentials for particles with spin, [19], [1], [5].

For all the examples known so far, the invariant modules have a particularly simple structure when expressed in a suitable coordinate system and gauge. In the one-dimensional case, they are the vector spaces of polynomials of degree less or equal than a positive integer n, and the corresponding

Lie algebra is simply the standard representation of $\mathfrak{sl}_2$ by first-order differential operators in one variable. It is easy to show that there are no other possibilities in one dimension, [13], [12]. In the case of quasi-exactly solvable differential operators in two variables associated to a complex Lie algebra, the modules also take the form of polynomial modules in the appropriate coordinates and gauge, [6]. They are either one of the vector spaces of bivariate polynomials $x^i y^j$ of bidegree (i,j) less or equal than an ordered pair (n,m) of positive integers, or one of the vector spaces of polynomials of total degree $i+j$ less or equal than a positive integer n, or one of the vector spaces of polynomials whose bidegree (i,j) satisfies a linear constraint of the form $i+rj \leq p$, for some positive integers r and p. We will refer to these modules as the rectangular, triangular or staircase modules in view of their structure within the lattice of bidegrees. The corresponding Lie algebras are given by the standard realizations of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, $\mathfrak{sl}_3$ and $\mathfrak{gl}_2 \ltimes \mathbb{C}^{r+1}$ by first-order differential operators in the plane. The inequivalent real forms of these algebras give rise to different quasi-exactly solvable potentials, [10]. It should be noted that while the rectangular and triangular modules are acted on irreducibly by $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{sl}_3$, this is *not* the case for the staircase module.

A fundamental question is to determine why practically every known example of partially solvable differential operator should be expressible as a polynomial in the generators of a finite-dimensional Lie algebra of first-order differential operators leaving the corresponding subspace invariant, and to what extent such a polynomial is unique. In particular, one would like to know if there exist modules for which the generic partially solvable differential operators are not Lie algebraic. These are the main problems that we address in this paper for the case of rectangular, triangular and staircase scalar or rank 2 modules. In the case in which the module is acted on irreducibly by the Lie algebra, which in our case would be the rectangular and triangular modules, a partial answer is provided by the Burnside Theorem, as pointed out by Turbiner, [24]. Recall that according to the Burnside Theorem, if V is a complex vector space and $\mathfrak{a}$ is a subalgebra of $\operatorname{End} V$ acting irreducibly, then every endomorphism of V can be expressed as a polynomial in the generators of $\mathfrak{a}$. This implies that any differential operator preserving a rectangular or a triangular module can be expressed as the sum of a polynomial in the generators of the realization of the corresponding Lie algebra and a differential operator annihilating the module. Note, however, that this argument does not provide any direct information on the relation between the order of a differential operator leaving a rectangular or triangu-

lar module invariant and the minimum degree of a polynomial representation thereof, owing to the presence of polynomial relations between the generators of the Lie algebra, which form a basis for the primitive ideal associated to the module. The case of the staircase modules is considerably more complicated, since the underlying Lie algebras of first-order differential operators do not act irreducibly, so that one cannot apply the Burnside Theorem. In fact, we will see that in stark contrast with the rectangular and triangular cases, the generic differential operator preserving the staircase module is *not* Lie-algebraic. One needs therefore to develop a different approach altogether. This is what we suceed to do in this paper, by developing a general graphical method which does not require the (scalar or rank 2) modules to be irreducible under the action of the corresponding algebra. We will thus entirely bypass the use of the Burnside Theorem and the need for extensive calculations in local coordinates. Indeed, the latter can get prohibitively difficult in higher dimensions. (Even the one-dimensional scalar case is quite involved when one works in coordinates, [22]). It is worth noting that the graphical method can be generalized without major difficulties to the case of rank k modules of polynomials in N variables. It is thus a powerful tool for future applications.

In Section 2, we briefly review the normal forms for the maximal Lie algebras of first-order differential operators in two complex variables which admit finite-dimensional modules of polynomials when expressed in a suitable coordinate system and gauge. In Section 3, we unravel by means of a general graphical method the Lie algebraic structure of the scalar differential operators of arbitrary finite order which preserve a rectangular, triangular or staircase module. We give a formula for the minimum degree of a polynomial in the generators of the corresponding Lie algebra representing the given quasi-exactly solvable differential operator. In the case of triangular modules, we also obtain an exact formula for the number of free parameters which determine the most general partially solvable differential operator of arbitrary finite order. Section 4 is concerned with 2×2 matrix differential operators of arbitrary finite order admitting the direct sum of two rectangular, triangular or staircase modules as an invariant subspace. For suitable values of the gap parameters, the underlying algebras of matrix differential operators give rise to new realizations of finite-dimensional Lie superalgebras of first-order differential operators. The abstract structure of these finite-dimensional Lie superalgebras is identified in every case. For the remaining values of the gap parameters, one obtains an infinite-dimensional Lie superalgebra. Remark-

ably, the graphical method introduced for scalar modules enables us in both the finite and infinite-dimensional cases to prove structure theorems for the corresponding partially solvable matrix differential operators.

2 Quasi-exactly solvable Lie algebras in $\mathbb{C}^2$

Our purpose in this section is to briefly recall from [6] the normal forms for the maximal Lie algebras of first-order differential operators in two complex variables admitting finite-dimensional modules of polynomials in suitable coordinates, along with the structure of these modules.

Let U be an open subset of $\mathbb{C}^2$, with local coordinates (x, y). The first-order differential operators on U,

$$T = f(x,y)\partial_x + g(x,y)\partial_y + h(x,y)\,,$$

with coefficients in $\mathrm{C}^\infty(U,\mathbb{C})$ form a Lie algebra which we will denote by $\mathfrak{D}^{(1)} = \mathfrak{D}^{(1)}(U)$. The space $\mathrm{C}^\infty(U,\mathbb{C})$ is naturally a $\mathfrak{D}^{(1)}$-module. We will use the terminology of [9], and refer to finite-dimensional subalgebras $\mathfrak{g}$ of $\mathfrak{D}^{(1)}$ admitting a finite-dimensional module $\mathcal{N} \subset \mathrm{C}^\infty(U,\mathbb{C})$ as *quasi-exactly solvable* Lie algebras. The local diffeomorphisms φ of U and the rescalings by smooth functions $e^\sigma \in \mathrm{C}^\infty(U,\mathbb{C}^*)$ define an infinite pseudogroup G of automorphisms Φ of $\mathfrak{D}^{(1)}$, whose action is given by

$$\Phi(T) = e^\sigma \cdot \varphi_* T \cdot e^{-\sigma}\,. \tag{2.1}$$

The classification under the action of G of all quasi-exactly solvable Lie algebras in two complex variables is known, [6] (its real counterpart has been recently completed, [10]). We shall focus our attention on the three maximal families of equivalence classes of quasi-exactly solvable Lie algebras whose normal forms preserve a module of polynomials.

i) $\mathfrak{g}^{11}_{n,m} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, with generators:

$$\begin{aligned} J^+_n &= x^2\partial_x - nx\,, \qquad J^- = \partial_x\,, \qquad J^0_n = x\partial_x - \frac{n}{2}\,,\\ K^+_m &= y^2\partial_y - my\,, \qquad K^- = \partial_y\,, \qquad K^0_m = y\partial_y - \frac{m}{2}\,, \end{aligned} \tag{2.2}$$

with $n, m \in \mathbb{Z}^+$, and associated finite-dimensional module given by

$$\mathcal{R}_{n,m} = \{x^i y^j \mid 0 \le i \le n\,,\ 0 \le j \le m\}\,.$$

ii) $\mathfrak{g}_n^{15} \simeq \mathfrak{sl}_3$, with generators:

$$J_n^1 = x^2\partial_x + xy\partial_y - nx\,, \qquad J_n^2 = xy\partial_x + y^2\partial_y - ny\,,$$
$$J^3 = y\partial_x\,, \qquad J^4 = \partial_x\,, \qquad J^5 = \partial_y\,, \qquad J^6 = x\partial_y\,,$$
$$J_n^7 = x\partial_x - \frac{n}{3}\,, \qquad J_n^8 = y\partial_y - \frac{n}{3}\,, \tag{2.3}$$

with $n \in \mathbb{Z}^+$ and corresponding finite-dimensional module given by

$$\mathcal{T}_n = \{x^i y^j \mid 0 \le i + j \le n\}\,.$$

iii) $\mathfrak{g}_p^{24,r} \simeq \mathfrak{gl}_2 \ltimes \mathbb{C}^{r+1}$, with generators:

$$J^1 = \partial_x\,, \qquad J_p^2 = x^2\partial_x + rxy\partial_y - px\,, \qquad J_p^3 = x\partial_x - \frac{p}{2}\,,$$
$$J^4 = y\partial_y\,, \qquad J^{5+i} = x^i\partial_y\,, \qquad 0 \le i \le r\,, \tag{2.4}$$

with $2 \le r \in \mathbb{Z}^+$, $p \in \mathbb{Z}^+$ and, for $q \in \mathbb{Z}^+$, associated module given by

$$\mathcal{S}_{p,q}^r = \{x^i y^j \mid 0 \le i + rj \le p\,,\ 0 \le j \le q\}\,,$$

As a special case, we shall consider the module $\mathcal{S}_p^r = \mathcal{S}_{p,[p/r]}^r$.

The nomenclature of the algebras is the same as the one used in [6], and the letters $\mathcal{R}$, $\mathcal{T}$ and $\mathcal{S}$ used to denote the modules reflect of course the rectangular, triangular and staircase structure of the lattices in the positive $\mathbb{Z}^+ \times \mathbb{Z}^+$ quadrant corresponding to the powers (i, j) that appear in $\mathcal{R}_{n,m}$, $\mathcal{T}_n$ and $\mathcal{S}_{p,q}^r$. The integers n, m, p, q which label the realizations of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, $\mathfrak{sl}_3$ and $\mathfrak{gl}_2 \ltimes \mathbb{C}^{r+1}$ in the list above are related to the global models for these algebras, where the underlying complex surfaces are respectively given by $\mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$, $\mathbb{P}_2(\mathbb{C})$ and the r-th Hirzebruch surface Σ_r, and the modules are sections of holomorphic line bundles over these surfaces, [8].

3 Scalar k-th order differential operators in $\mathbb{C}^2$ with invariant modules of polynomials

Our aim in this section is to prove structure theorems for the subalgebras of the associative algebra $\mathfrak{D} = \mathfrak{D}(U)$ of scalar differential operators in two

complex variables

$$T = \sum_{0\le i,j<\infty} f_{ij}(x,y)\,\partial_x^i\partial_y^j\,, \qquad f_{ij}(x,y)\in \mathrm{C}^\infty(U,\mathbb{C})\,, \tag{3.1}$$

which map the modules $\mathcal{R}_{n,m}$, $\mathcal{T}_n$ and $\mathcal{S}^r_{p,q}$ to themselves. Let us remark that our results will also be valid when x and y are restricted to the real plane $\mathbb{R}^2$. The motivation for our study arises from a theorem of Turbiner concerning scalar k-th order differential operators in one variable, [22]. To set Turbiner's theorem in context, let us recall, [13], [12], that up to equivalence under the one-dimensional analog to the pseudogroup G defined in (2.1), there is a unique family of quasi-exactly solvable Lie algebras of first-order differential operators on the line, namely the family of representations of $\mathfrak{sl}_2$ given by $\mathfrak{g}_n = \{J_n^+,\, J^-,\, J_n^0\}$, where

$$J_n^+ = x^2\partial_x - nx\,, \qquad J^- = \partial_x\,, \qquad J_n^0 = x\partial_x - \frac{n}{2}\,. \tag{3.2}$$

The associated invariant module is the polynomial module

$$\mathcal{P}_n = \{x^i \mid 0\le i\le n\}\,.$$

Let $\tilde{\mathfrak{D}}$ be the associative algebra of all linear differential operators in one complex variable, and let $\tilde{\mathfrak{D}}^{(k)}$ denote its subspace of differential operators of order at most k. We shall denote by $\tilde{\mathfrak{D}}^{(>k)}$ the subalgebra of $\tilde{\mathfrak{D}}$ spanned by all differential operators of the form $T\,\partial_x^{k+1}$, where T is an arbitrary differential operator. Let $\mathfrak{P}_n$ be the subalgebra of $\tilde{\mathfrak{D}}$ of differential operators which preserve $\mathcal{P}_n$, and let $\mathfrak{P}_n^{(k)} = \mathfrak{P}_n\cap\tilde{\mathfrak{D}}^{(k)}$. Finally, let $\mathcal{S}^k(\mathfrak{sl}_2) = \bigoplus_{i=0}^k S^i(\mathfrak{sl}_2)$ be the direct sum of the first k symmetric powers of $\mathfrak{sl}_2$. Turbiner's theorem may be reformulated as follows:

Theorem 3.1 *i) The linear map $\rho_n^k : \mathcal{S}^k(\mathfrak{sl}_2)\to\mathfrak{P}_n^{(k)}$ determined by the representation $\rho_n : \mathfrak{sl}_2\to\mathfrak{g}_n$ is surjective for all $k=0,\dots,n$. ii) If $k>n$, then $\mathfrak{P}_n^{(k)} = \mathfrak{P}_n^{(n)}\oplus(\tilde{\mathfrak{D}}^{(k)}\cap\tilde{\mathfrak{D}}^{(>n)})$.*

Remark. Turbiner's original proof was done by explicit computation. We present here a simplified proof whose principle will serve as a guide in the higher dimensional case.

Proof. We first note that $\ker\pi_n = \tilde{\mathfrak{D}}^{(>n)}$, where π_n denotes the induced homomorphism $\pi_n : \mathfrak{P}_n\to\mathrm{End}(\mathcal{P}_n)$, so *ii)* clearly holds. Therefore, the

restriction of π_n to $\mathfrak{P}_n^{(n)}$ is injective, so $\dim \mathfrak{P}_n^{(n)} \leq (n+1)^2$. Now, the monomials $\{(J_n^{\pm})^i (J_n^0)^{j-i}\}_{i=0}^{j}$, $j = 0, 1, \ldots$, are all linearly independent differential operators. In particular, there are $(n+1)^2$ monomials of order at most n, so they form a basis of $\mathfrak{P}_n^{(n)}$. Since $(J_n^{\pm})^i (J_n^0)^{j-i} = x^{j \pm i} \partial_x^j + \cdots$, no monomial of degree higher than k may appear in the expansion of a differential operator $T^{(k)} \in \mathfrak{P}_n^{(k)}$. Q.E.D.

Thus, Theorem 3.1 simply expresses the fact that any differential operator

$$T = \sum_{i=0}^{k \leq n} f_i(x)\, \partial_x^i\,,$$

preserving $\mathcal{P}_n$ may be written as a polynomial of degree k in the generators (3.2).

Later, Turbiner proposed a proof of Theorem 3.1 based on the irreducibility of the module $\mathcal{P}_n$ under the action of $\mathfrak{g}_n$ and the Burnside Theorem. However, this argument only guarantees that $T^{(k)}$ will be the sum of a polynomial in the generators of $\mathfrak{g}_n$ with an operator annihilating the module. In particular, it gives no information on the degree of the polynomial.

The linear map $\rho_n^k : \mathcal{S}^k(\mathfrak{sl}_2) \to \mathfrak{P}_n^{(k)} \subset \tilde{\mathfrak{D}}^{(k)}$ is clearly not injective for $k \geq 2$. In fact, the set $\{(J_n^{\pm})^i (J_n^0)^{j-i}\}_{i=0}^{j}$, $j = 0, \ldots, k$ is a basis of $\operatorname{im} \rho_n^k$ (the completeness follows from the quadratic relation $J_n^+ J^- = J_n^0 J_n^0 - J_n^0 - \frac{1}{4} n(n+2)$ which expresses the scalar action of the Casimir operator on $\mathcal{P}_n$). Thus,

$$\dim(\ker \rho_n^k) = \dim \mathcal{S}^k(\mathfrak{sl}_2) - (k+1)^2 = \frac{1}{6}(k+1)k(k-1)\,. \tag{3.3}$$

The dimension of the kernel of the induced map $\bar{\rho}_n^k : S^k(\mathfrak{sl}_2) \to \tilde{\mathfrak{D}}^{(k)} / \tilde{\mathfrak{D}}^{(k-1)}$ may also be easily obtained:

$$\dim(\ker \bar{\rho}_n^k) = \frac{1}{2} k(k-1)\,. \tag{3.4}$$

We emphasize that equations (3.3) and (3.4) are still valid if the cohomology parameter n labeling the representation $\mathfrak{g}_n$ is replaced by an arbitrary complex number λ; see [15] and [4] for more details.

We now focus on the two-variable modules $\mathcal{R}_{n,m}$, $\mathcal{T}_n$ and $\mathcal{S}_{p,q}^r$, and define $\mathfrak{R}_{n,m}$, $\mathfrak{T}_n$ and $\mathfrak{S}_{p,q}^r$ as the subalgebras of $\mathfrak{D}$ which preserve these modules. Let

$\pi_{n,m} : \mathfrak{R}_{n,m} \to \mathrm{End}(\mathcal{R}_{n,m})$, $\pi_n : \mathfrak{T}_n \to \mathrm{End}(\mathcal{T}_n)$, and $\pi^r_{p,q} : \mathfrak{S}^r_{p,q} \to \mathrm{End}(\mathcal{S}^r_{p,q})$ denote the induced homomorphisms of associative algebras. We have the following elementary result:

Lemma 3.2 *The kernels of $\pi_{n,m}$, π_n, and $\pi^r_{p,q}$ are given by*

$$\ker \pi_{n,m} = \Big\{ T \in \mathfrak{D} \mid T = \sum_{\substack{i>n \text{ or} \\ j>m}} f_{ij}(x,y)\, \partial_x^i \partial_y^j \Big\}\,,$$

$$\ker \pi_n = \Big\{ T \in \mathfrak{D} \mid T = \sum_{i+j>n} f_{ij}(x,y)\, \partial_x^i \partial_y^j \Big\}\,,$$

$$\ker \pi^r_{p,q} = \Big\{ T \in \mathfrak{D} \mid T = \sum_{\substack{i+rj>p \\ \text{or } j>q}} f_{ij}(x,y)\, \partial_x^i \partial_y^j \Big\}\,.$$

We now choose the following distinguished complements to the kernels in Lemma 3.2:

$$\tilde{\mathfrak{R}}_{n,m} = \Big\{ T \in \mathfrak{R}_{n,m} \mid T = \sum_{\substack{0\le i\le n \\ 0\le j\le m}} f_{ij}(x,y)\, \partial_x^i \partial_y^j \Big\}\,,$$

$$\tilde{\mathfrak{T}}_n = \Big\{ T \in \mathfrak{T}_n \mid T = \sum_{0\le i+j\le n} f_{ij}(x,y)\, \partial_x^i \partial_y^j \Big\}\,,$$

$$\tilde{\mathfrak{S}}^r_{p,q} = \Big\{ T \in \mathfrak{S}^r_{p,q} \mid T = \sum_{\substack{0\le i+rj\le p \\ 0\le j\le q}} f_{ij}(x,y)\, \partial_x^i \partial_y^j \Big\}\,.$$

Note that the order of a differential operator in the distinguished complement $\tilde{\mathfrak{S}}^r_{p,q}$ cannot exceed p. The next result follows from a simple constructive argument:

Lemma 3.3 *The restrictions $\tilde{\pi}_{n,m} : \tilde{\mathfrak{R}}_{n,m} \to \mathrm{End}(\mathcal{R}_{n,m})$, $\tilde{\pi}_n : \tilde{\mathfrak{T}}_n \to \mathrm{End}(\mathcal{T}_n)$, and $\tilde{\pi}^r_{p,q} : \tilde{\mathfrak{S}}^r_{p,q} \to \mathrm{End}(\mathcal{S}^r_{p,q})$ are vector space isomorphisms.*

Proof. Let us prove the lemma for one of the modules, say $\mathcal{T}_n$. Let $\mathfrak{I} \in \mathrm{End}(\mathcal{T}_n)$. The differential operator $T \in \tilde{\mathfrak{T}}_n$ with coefficients f_{ij} given by

$$f_{00} = \mathfrak{I}(1)\,, \qquad f_{10} = \mathfrak{I}(x) - x f_{00}\,, \qquad f_{01} = \mathfrak{I}(y) - y f_{00}\,, \ldots$$

clearly satisfies $\tilde{\pi}_n(T) = \mathfrak{I}$. Q.E.D.

Note that the coefficients f_{ij} of the differential operators in the distinguished complements are polynomials in x and y. The subalgebra $\mathfrak{D}_P$ of $\mathfrak{D}$ of differential operators (3.1) with polynomial coefficients f_{ij} inherits a natural $\mathbb{Z} \times \mathbb{Z}$ grading from $\mathbb{C}[x, y]$. We shall say that a differential operator $T \in \mathfrak{D}_P$ has *bidegree* $\deg T = (i, j)$ if $T(x^{i_0} y^{j_0}) \in \langle x^{i_0+i} y^{j_0+j} \rangle$ for all $(i_0, j_0) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. Note that the generators of $\mathfrak{g}^{11}_{n,m}$, $\mathfrak{g}^{15}_n$ and $\mathfrak{g}^{24,r}_p$ all have well-defined bidegree, [23]. A set of differential operators in $\mathfrak{D}_P$ with different bidegrees is of course linearly independent.

We will obtain in what follows structure theorems for each of the distinguished complements. Let us remark that both $\mathfrak{g}^{11}_{n,m}$ and $\mathfrak{g}^{15}_n$ act irreducibly on their associated modules $\mathcal{R}_{n,m}$ and $\mathcal{T}_n$. Therefore, a direct application of the Burnside Theorem shows that any differential operator T in $\tilde{\mathfrak{R}}_{n,m}$ (respectively $\tilde{\mathfrak{T}}_n$) may be constructed as a polynomial in the generators of $\mathfrak{g}^{11}_{n,m}$ (respectively $\mathfrak{g}^{15}_n$) plus an element of $\ker \pi_{n,m}$ (respectively $\ker \pi_n$), [24]. Note that the action of $\mathfrak{g}^{24,r}_p$ on $\mathcal{S}^r_{p,q}$ is *neither* irreducible *nor* completely reducible, contrary to the the assertion in ref. [23]. In fact, we will exhibit differential operators in $\tilde{\mathfrak{S}}^r_{p,q}$ which cannot be expressed as a polynomial in the generators of $\mathfrak{g}^{24,r}_p$ plus an element in $\ker \pi^r_{p,q}$.

3.1 The rectangular module $\mathcal{R}_{n,m}$

Let $\mathfrak{D}^{(k,l)}$ denote the subspace of $\mathfrak{D}$ spanned by all differential operators of x-order at most k and y-order at most l, and let $\tilde{\mathfrak{R}}^{(k,l)}_{n,m} = \tilde{\mathfrak{R}}_{n,m} \cap \mathfrak{D}^{(k,l)}$. The structure theorem for $\tilde{\mathfrak{R}}_{n,m}$ is a straightforward generalization of Theorem 3.1:

Theorem 3.4 *Let $\bar{\rho}^i_n(x) : S^i(\mathfrak{sl}_2) \to \mathfrak{P}_n(x)$ and $\bar{\rho}^j_m(y) : S^j(\mathfrak{sl}_2) \to \mathfrak{P}_m(y)$ be the linear maps determined by $\rho_n(x) : \mathfrak{sl}_2 \to \mathfrak{g}_n(x)$ and $\rho_m(y) : \mathfrak{sl}_2 \to \mathfrak{g}_m(y)$, respectively. Then,*

$$\bigoplus_{\substack{0 \le i \le k \\ 0 \le j \le l}} \bar{\rho}^i_n(x) \otimes \bar{\rho}^j_m(y) : \bigoplus_{\substack{0 \le i \le k \\ 0 \le j \le l}} S^i(\mathfrak{sl}_2) \otimes S^j(\mathfrak{sl}_2) \to \tilde{\mathfrak{R}}^{(k,l)}_{n,m} ,$$

is surjective for all $k = 0, \ldots, n$ and $l = 0, \ldots, m$.

3.2 The triangular module $\mathcal{T}_n$

Let $\mathfrak{D}^{(k)}$ be the subspace of $\mathfrak{D}$ spanned by all differential operators of order at most k, and let $\tilde{\mathfrak{T}}^{(k)}_n = \tilde{\mathfrak{T}}_n \cap \mathfrak{D}^{(k)}$. As before, let $\mathcal{S}^k(\mathfrak{sl}_3) = \bigoplus_{i=0}^k S^i(\mathfrak{sl}_3)$ be the

direct sum of the first k symmetric powers of $\mathfrak{sl}_3$, and let $\rho_n^k : \mathcal{S}^k(\mathfrak{sl}_3) \to \mathfrak{D}^{(k)}$ be the linear map determined by the representation $\rho_n : \mathfrak{sl}_3 \to \mathfrak{g}_n^{15}$. Our purpose is to construct a suitable basis of $\operatorname{im}\rho_n^k$ for all $k \in \mathbb{Z}^+$, from which the structure theorem for $\tilde{\mathfrak{T}}_n$ will arise as a simple corollary. The bidegree of the generators (2.3) of $\mathfrak{g}_n^{15}$ is given by:

$$\begin{aligned} &\deg J_n^1 = (1,0)\,, \quad \deg J_n^2 = (0,1)\,, \quad \deg J^3 = (-1,1)\,, \quad \deg J^4 = (-1,0)\,, \\ &\deg J^5 = (0,-1)\,, \quad \deg J^6 = (1,-1)\,, \quad \deg J_n^7 = (0,0)\,, \quad \deg J_n^8 = (0,0)\,, \end{aligned}$$

and the bidegree of a monomial $J_n^K = (J_n^1)^{k_1} \cdots (J_n^8)^{k_8}$ is simply $\sum_{i=0}^{6} k_i \deg J_n^i$. Let $|K| = k_1 + \cdots + k_8$ denote the *degree* of J_n^K. Note that $|K|$ is just the order of J_n^K. We define the *length* of a monomial J_n^K as

$$|J_n^K| = \min_{\deg J_n^L = \deg J_n^K} |L|\,,$$

where the minimum is taken over the set of monomials of degree at most $|K|$. In other words, if $\deg J_n^K = (i,j)$, then $|J_n^K|$ is just the minimum number of "steps" required to map a point $(i_0, j_0) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ to $(i_0 + i, j_0 + j)$. A monomial J_n^K is of *maximal length* if $|J_n^K| = |K|$. The monomials $1, J_n^1, \dots, J_n^6$ are obviously of maximal length. The following facts follow from an easy graphical argument, cf. Fig. 1:

i) There is exactly one monomial of maximal length of bidegree (i,j) for each $(i,j) \in \mathbb{Z} \times \mathbb{Z}$, up to the ordering of its factors.

ii) The monomials of maximal length l are $\{(J_n^{s-1})^j\,(J_n^s)^{l-j}\}_{j=1-\delta_{l0}}^{l}$, with $s = 1, \dots, 6$ and $J_n^0 = J_n^6$.

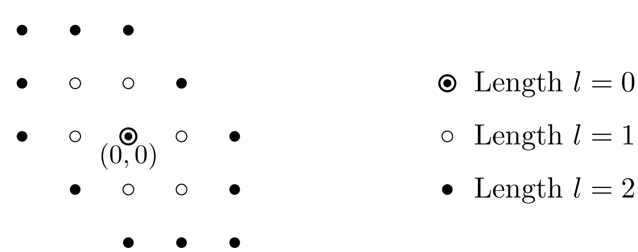

Figure 1: Lattice of bidegrees of monomials of length $l \leq 2$.

Note that there are $6l$ monomials of maximal length $l \geq 1$. If J_n^L is a monomial of maximal length, the monomials $J_n^L\,(J_n^7)^i\,(J_n^8)^j$, $i, j \geq 0$, are

linearly independent, and have the same bidegree as J_n^L. Remarkably, the monomials of this form are also a generating set of $\operatorname{im}\rho_n^k$ for all $k \in \mathbb{Z}^+$, according to the following lemma:

Lemma 3.5

i) The monomials

$$\left\{ (J_n^{s-1})^j (J_n^s)^{l-j} (J_n^7)^i (J_n^8)^{t-l-i} \,\middle|\, 1-\delta_{l0} \le j \le l \le t \le k,\ 0 \le i \le t-l \right\}, \tag{3.5}$$

with $s = 1, \dots, 6$ *and* $J_n^0 = J_n^6$*, form a basis of* $\operatorname{im}\rho_n^k$ *for all* k.

ii) The number of k*-th order monomials in the basis is* $n_k = (k+1)^3$*, and*

$$\dim(\operatorname{im}\rho_n^k) = \frac{1}{4}(k+1)^2(k+2)^2. \tag{3.6}$$

Proof. Linear independence of the monomials follows from the preceeding remarks. Completeness is a consequence of the following 9 quadratic relations,

$$\begin{aligned}
&J_n^1 J^3 = J_n^2\big(J_n^7 + \frac{n}{3}\big)\,, \qquad J_n^1 J^4 = (J_n^7)^2 + J_n^7 J_n^8 - J_n^7 + \frac{n}{3} J_n^8 - \frac{n}{3}(\frac{n}{3}+1)\,,\\
&J_n^1 J^5 = J^6\big(J_n^7 + J_n^8 - (\frac{n}{3}+1)\big)\,, \qquad J_n^2 J^4 = J^3\big(J_n^7 + J_n^8 - (\frac{n}{3}+1)\big)\,,\\
&J_n^2 J^5 = J_n^7 J_n^8 + (J_n^8)^2 + \frac{n}{3} J_n^7 - J_n^8 - \frac{n}{3}(\frac{n}{3}+1)\,, \qquad J_n^2 J^6 = J_n^1\big(J_n^8 + \frac{n}{3}\big)\,,\\
&J^3 J^5 = J^4\big(J_n^8 + \frac{n}{3}\big)\,, \qquad J^3 J^6 = J_n^7 J_n^8 + \frac{n}{3} J_n^7 + (\frac{n}{3}+1) J_n^8 + \frac{n}{3}(\frac{n}{3}+1)\,,\\
&J^4 J^6 = J^5\big(J_n^7 + \frac{n}{3} + 1\big)\,,
\end{aligned} \tag{3.7}$$

which allow us to reduce any monomial J_n^K to a linear combination of the elements of the set (3.5). The second assertion now follows by straightforward computation. Q.E.D.

From Lemma 3.5, we obtain the following result:

Theorem 3.6 *The linear map* $\rho_n^k : \mathcal{S}^k(\mathfrak{sl}_3) \to \tilde{\mathfrak{T}}_n^{(k)}$ *determined by the representation* $\rho_n : \mathfrak{sl}_3 \to \mathfrak{g}_n^{15}$ *is surjective for all* $k = 0, \dots, n$.

Proof. The map $\rho_n^n : \mathcal{S}^n(\mathfrak{sl}_3) \to \tilde{\mathfrak{T}}_n$ is clearly surjective, since $\dim \tilde{\mathfrak{T}}_n = \frac{1}{4}(n+1)^2(n+2)^2$. The highest order terms of the monomials in (3.5) are all

independent, which implies that no monomial of degree higher than k may appear in the expansion of a differential operator $T^{(k)} \in \tilde{\mathfrak{T}}_n^{(k)}$. Q.E.D.

Theorem 3.6 is stated in a different form in [23]. It should be noted, however, that just as in the $\mathfrak{sl}_2$ case, this theorem does not follow directly from the Burnside Theorem, contrary to what is stated in [23]. As another immediate consequence of Lemma 3.5, we have:

Corollary 3.7 *The kernel of the homomorphism $\bar{\rho}_n : \mathfrak{U}(\mathfrak{sl}_3) \to \mathfrak{D}$ determined by the representation $\rho_n : \mathfrak{sl}_3 \to \mathfrak{g}_n^{15}$ is the primitive ideal generated by the quadratic relations* (3.7)*.*

Let us emphasize that these results are completely independent of the discrete character of the cohomology parameter n labeling the Lie algebras $\mathfrak{g}_n^{15}$, being also valid for a complex cohomology parameter λ, as considered in [4] and [15]. The graphical method used to obtain the basis (3.5) can be generalized to the N-dimensional representations of $\mathfrak{sl}_{N+1}$ which appear as the hidden symmetry algebra of the Calogero model and its quasi-exactly solvable extensions, [3], [17], [14]. Finally, let us give the formula analogous to (3.4) for the map $\bar{\rho}_n^k : S^k(\mathfrak{sl}_3) \to \mathfrak{D}^{(k)}/\mathfrak{D}^{(k-1)}$:

$$\dim(\ker \bar{\rho}_n^k) = \frac{(k+1)k(k-1)}{7!}\big(k^4 + 28k^3 + 323k^2 + 1988k + 2052\big)\,.$$

3.3 The staircase module $\mathcal{S}_{p,q}^r$

The structure of the space of operators preserving this type of modules is considerably more complicated than that of the other two modules. As remarked before, the action of $\mathfrak{g}_p^{24,r}$ on $\mathcal{S}_{p,q}^r$ is reducible, leading to a number of interesting consequences.

The bidegrees of the generators of $\mathfrak{g}_p^{24,r}$ are:

$$\begin{aligned} &\deg J^1 = (-1,0)\,, \qquad \deg J_p^2 = (1,0)\,, \qquad \deg J_p^3 = (0,0)\,, \\ &\deg J^4 = (0,0)\,, \qquad \deg J^{5+i} = (i,-1)\,, \qquad 0 \le i \le r\,. \end{aligned}$$

We shall refer to the second component of the bidegree as the y-degree. Note that the y-degree of the generators (2.4) of $\mathfrak{g}_p^{24,r}$ is nonpositive. Thus, no operator in the distinguished complement $\tilde{\mathfrak{S}}_{p,q}^r$ with positive y-degree (as $T = y\partial_x^2$, if $r = 2$, $p \ge 2$ and $q = [p/2]$), can be obtained as a polynomial

in the generators plus an operator in $\ker \pi^r_{p,q}$. Moreover, not every operator in $\tilde{\mathfrak{S}}^r_{p,q}$ with nonpositive y-degree may be written as a polynomial in the generators of $\mathfrak{g}^{24,r}_p$ only. An example is given by $T = x^2(x^2\partial_x^2 - 2x\partial_x - 2y\partial_y + 2)$ in the case $p = 2q = r = 2$. Our next objective is to prove that every k-th order differential operator $T^{(k)}$ in $\tilde{\mathfrak{S}}^{r\downarrow}_{p,q}$, the subspace of $\tilde{\mathfrak{S}}^r_{p,q}$ of differential operators with nonpositive y-degree, may be written as the sum of a k-th degree polynomial in the generators of $\mathfrak{g}^{24,r}_p$ and an operator in $\ker \pi^r_{p,q}$. We shall construct a suitable collection of monomials in the generators (2.4) such that their projections in $\tilde{\mathfrak{S}}^r_{p,q}$ along $\ker \pi^r_{p,q}$ form a basis of $\tilde{\mathfrak{S}}^{r\downarrow}_{p,q}$. (Note that in the triangular case every monomial of degree at most n has a non-trivial projection in $\tilde{\mathfrak{T}}_n$ along $\ker \pi_n$). We follow the approach of the previous section, obtaining first a set of monomials of maximal length (in this case the bidegree does not determine uniquely the monomials of maximal length), and then extend it to a basis of the image of the map $\rho^{r,k}_p : \mathcal{S}^k(\mathfrak{gl}_2 \ltimes \mathbb{C}^{r+1}) \to \mathfrak{D}^{(k)}$ determined by the representation $\rho^r_p : \mathfrak{gl}_2 \ltimes \mathbb{C}^{r+1} \to \mathfrak{g}^{24,r}_p$.

Lemma 3.8

i) The monomials[1]

$$\left\{(J_p^{1+\epsilon})^{l-j}(J^{5+\epsilon r})^j(J_p^3)^n(J^4)^{t-l-n},\ (J^5)^s J^{5+i}(J^{5+r})^{l-s-1}(J_p^3)^n(J^4)^{t-l-n}\right\},$$

with $\epsilon = 0, 1$, $0 \le j \le l \le t \le k$, $0 \le s \le l-1$, $0 \le n \le t-l$, *and* $1 \le i \le r - \delta_{s0}$ *form a basis of* $\operatorname{im} \rho^{r,k}_p$ *for all* k.

ii) The number of k*-th order monomials in the basis is*

$$n_k = \frac{1}{6}(k+1)(k+2)\big((r+2)k+3\big)\,.$$

The basis described in the previous lemma does not completely fulfill our requirements, for it contains monomials of order less than p acting trivially in $\mathcal{S}^r_{p,q}$. A subset of monomials acting non-trivially can be constructed by a simple graphical argument, cf. Fig. 2.

Lemma 3.9 *The projections in* $\tilde{\mathfrak{S}}^r_{p,q}$ *along* $\ker \pi^r_{p,q}$ *of the monomials*

$$\left\{(J_p^{1+\epsilon})^s(J^{5+\epsilon r})^j(J_p^3)^n(J^4)^m,\ (J^5)^t J^{5+i}(J^{5+r})^{j-t-1}(J_p^3)^n(J^4)^m\right\}, \tag{3.8}$$

[1]By convention, the second group of monomials is not present if $l = 0$.

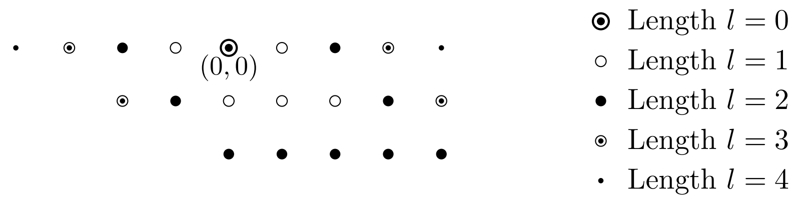

Figure 2: Lattice of bidegrees of monomials acting non-trivially in $\mathcal{S}^2_{4,2}$

with $\epsilon = 0, 1,\ 0 \le j \le q,\ 0 \le s \le p - jr,\ 0 \le t \le j - 1,\ 1 \le i \le r - \delta_{t0},$ $0 \le n,\ 0 \le m \le q - j,$ *and* $n + rm \le p - \tilde{s} - jr$ *(where* $\tilde{s} = s$ *for the first group of monomials, and* $\tilde{s} = 0$ *for the second group), form a basis of* $\tilde{\mathfrak{S}}^{r\downarrow}_{p,q}$.

Proof. The projections of the monomials in $\tilde{\mathfrak{S}}^r_{p,q}$ along $\ker \pi^r_{p,q}$ are certainly linearly independent. The number of monomials in (3.8) is given by

$$N^r_{pq} = \sum_{j=0}^{q} \sum_{s=0}^{p-jr} f(j,s)\sigma\big(r, p - s - jr, \min\big(q, [\frac{p-s}{r}]\big) - j\big)\,,$$

where $f(j,s) = 2$ if $s \ge 1$ and $f(j,0) = jr+1$, and $\sigma(r_0, p_0, q_0) = \dim \mathcal{S}^{r_0}_{p_0,q_0} = (q_0+1)(p_0+1-\frac{1}{2}r_0q_0)$. We leave as an exercise to the reader to verify that

$$N^r_{pq} = \dim \tilde{\mathfrak{S}}^{r\downarrow}_{p,q} = \frac{(q+1)(q+2)}{2}\Big((p+1)(p+1-qr) + \frac{qr^2}{12}(3q+1)\Big)\,. \quad \text{Q.E.D.}$$

The structure theorem for $\tilde{\mathfrak{S}}^{r\downarrow}_{p,q}$ now follows from the fact that the highest order derivatives of the monomials (3.8) are all independent.

Theorem 3.10 *Let* $T^{(k)}$ *be a* k*-th order differential operator in* $\tilde{\mathfrak{S}}^{r\downarrow}_{p,q}$*. If* $k \le q$*, then* $T^{(k)}$ *may be represented as a* k*-th degree polynomial in the generators of* $\mathfrak{g}^{24,r}_p$*. If* $k > q$*, then* $T^{(k)}$ *may be expressed as the projection of such a polynomial in* $\tilde{\mathfrak{S}}^r_{p,q}$ *along* $\ker \pi^r_{p,q}$.

We shall now restrict ourselves to the "untruncated" modules $\mathcal{S}^r_p = \mathcal{S}^r_{p,[p/r]}$ and the distinguished spaces of differential operators $\tilde{\mathfrak{S}}^r_p = \tilde{\mathfrak{S}}^r_{p,[p/r]}$ which preserve them. We are interested in obtaining a method for constructing the most general differential operator in $\tilde{\mathfrak{S}}^r_p$ irrespective of their y-th degree. Following [23], we introduce a well-adapted grading for the differential operators in $\tilde{\mathfrak{S}}^r_p$. If $T \in \tilde{\mathfrak{S}}^r_p$ has a well-defined bidegree (i,j), we define its *total degree* as $\operatorname{Deg} T = i + rj$. The constructive method outlined in the proof of Lemma 3.3

allows us to obtain the differential operator T representing any given endomorphism of $\mathcal{S}_p^r$. However, it does not give any information on the order of T, which limits its value in quantum mechanical applications. The following elementary lemma partially describes the subspace $\tilde{\mathfrak{S}}_p^{r,(k)} = \tilde{\mathfrak{S}}_p^r \cap \mathfrak{D}^{(k)}$ of k-th order differential operators.

Lemma 3.11 *Let $T^{(k)}$ be an element of $\tilde{\mathfrak{S}}_p^{r,(k)}$ homogeneous with respect to the total degree. Then $-kr \leq \operatorname{Deg} T^{(k)} \leq k$.*

Proof. The lower bound is obvious, for $T^{(k)}$ is a linear differential operator with polynomial coefficients. Now, if $\operatorname{Deg} T^{(k)} = d > k$, $T^{(k)}$ must annihilate the last d diagonals $\{x^i y^j \mid i + rj = p - d + 1, \dots, p\}$, but a (nonzero) k-th order differential operator cannot annihilate more than k diagonals. Q.E.D.

We can use this result and Theorem 3.10 to construct the most general differential operator in $\tilde{\mathfrak{S}}_p^{r,(k)}$. As an application, we give the explicit form of the most general differential operator $T^{(2)}$ in $\tilde{\mathfrak{S}}_p^{2,(2)}$, which may be used to construct partially solvable Schrödinger operators in two variables by solving the corresponding equivalence problem, [9]. We assume that $p \geq 4$ (otherwise some of the terms would just have a non-vanishing projection on $\ker \pi_p^2$ along $\tilde{\mathfrak{S}}_p^2$). Then, $T^{(2)} = \sum_{i=-4}^{2} T_i$, where:

$$
\begin{aligned}
T_{-4} &= a_1\, \partial_y^2 \,, \\
T_{-3} &= a_2\, \partial_x \partial_y + a_3\, x\partial_y^2 \,, \\
T_{-2} &= a_4\, \partial_x^2 + a_5\, x\partial_x\partial_y + a_6\, x^2\partial_y^2 + a_7\, y\partial_y^2 + a_8\, \partial_y \,, \\
T_{-1} &= a_9\, x\partial_x^2 + a_{10}\, x^2\partial_x\partial_y + a_{11}\, y\partial_x\partial_y + a_{12}\, x^3\partial_y^2 + a_{13}\, xy\partial_y^2 + a_{14}\, \partial_x + a_{15}\, x\partial_y \,, \\
T_0 &= a_{16}\, x^2\partial_x^2 + a_{17}\, y\partial_x^2 + a_{18}\, x^3\partial_x\partial_y + a_{19}\, xy\partial_x\partial_y + a_{20}\, x^4\partial_y^2 + a_{21}\, x^2y\partial_y^2 \\
&\quad + a_{22}\, y^2\partial_y^2 + a_{23}\, x\partial_x + a_{24}\, x^2\partial_y + a_{25}\, y\partial_y + a_{26} \,, \\
T_1 &= \big(a_{27}\, x^2\partial_x + a_{28}\, y\partial_x + a_{29}\, x^3\partial_y + a_{30}\, xy\partial_y + a_{31}\, x\big)(x\partial_x + 2y\partial_y - p) \,, \\
T_2 &= \big(a_{32}\, x^2 + a_{33}\, y\big)(x\partial_x + 2y\partial_y - p + 1)(x\partial_x + 2y\partial_y - p) \,.
\end{aligned}
$$

Note that $T_{-4}, \dots, T_0$ are linear combinations of monomials, reflecting the fact that their total degree is nonpositive (an operator of this type is sometimes referred to in the literature as *exactly solvable* since it preserves an infinite flag of polynomial subspaces). Note that the terms with coefficients a_{17}, a_{28} and a_{33} have positive y-degree, so they *cannot* be obtained from a polynomial in the generators of $\mathfrak{g}_p^{24,2}$. This is one of the first examples of a

partially solvable differential operator preserving a finite-dimensional space of polynomials which is not Lie algebraic. (A number of partially solvable operators which preserve a finite-dimensional module of monomials but are not Lie algebraic are also discussed in [16]. The exponents of the monomials spanning these modules do not form a continuous chain $\{0, 1, \dots, n\}$).

4 Matrix k-th order differential operators in $\mathbb{C}^2$ with invariant modules of polynomials

We shall now consider the more general case of k-th order 2×2 matrix differential operators admitting finite-dimensional invariant subspaces. Our goal is to prove structure theorems for these operators, which are the counterparts of Theorems 3.4, 3.6 and 3.10, in the case in which the invariant rank 2 module $\boldsymbol{\mathcal{N}} \subset \mathrm{C}^\infty(U, \mathbb{C} \oplus \mathbb{C})$ is either a direct sum of rectangular, triangular or staircase modules.

Consider the associative algebra $\mathfrak{D} = \mathfrak{D}(U)$ of all 2×2 matrix differential operators with smooth coefficient functions. We introduce in $\mathfrak{D}$ the usual $\mathbb{Z}_2$-grading; an operator

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{4.1}$$

is said to be *even* if $T_{12} = T_{21} = 0$, and *odd* if $T_{11} = T_{22} = 0$. The associative algebra $\mathfrak{D}$ becomes then a Lie superalgebra with generalized Lie product given by

$$[T_1, T_2]_s = T_1 T_2 - (-1)^{\deg T_1 \deg T_2} T_2 T_1 \,. \tag{4.2}$$

(We shall use the symbol $\mathfrak{D}$ to denote both the associative algebra and the Lie superalgebra). We will construct graded subalgebras of $\mathfrak{D}$ of matrix differential operators preserving the above mentioned vector-valued modules, in analogy with the scalar case.

4.1 The rectangular module $\mathcal{R}_{n_1,m_1} \oplus \mathcal{R}_{n_2,m_2}$

This case has been recently studied in [2]. We shall be very concise, for the results are a natural generalization of the single-variable case considered in [1] and [5]. We shall assume that $n = n_2 \geq n_1$ and $m = m_1 \geq m_2$ (the other

choice $m_2 \geq m_1$ leads to analogous results). Let $\Delta = n - n_1$ and $\Gamma = m - m_2$. Let us denote the direct sum $\mathcal{R}_{n-\Delta,m} \oplus \mathcal{R}_{n,m-\Gamma}$ by $\boldsymbol{\mathcal{R}}^{n-\Delta,m}_{n,m-\Gamma}$. Consider the graded subalgebra $\mathfrak{s}^{\Delta,\Gamma}_{n,m}$ of $\boldsymbol{\mathfrak{D}}$ generated by the $8 + 2(\Delta+1)(\Gamma+1)$ matrix differential operators given by

$$
\begin{aligned}
&S^\epsilon = \operatorname{diag}(J^\epsilon_{n-\Delta}, J^\epsilon_n)\,, && T^\epsilon = \operatorname{diag}(K^\epsilon_m, K^\epsilon_{m-\Gamma})\,, \qquad \epsilon = \pm, 0\,,\\
&J = \frac{1}{2}\operatorname{diag}(n+\Delta, n)\,, && K = \frac{1}{2}\operatorname{diag}(m, m+\Gamma)\,,\\
&Q^-_{\alpha\beta} = q_\alpha(x)\,\bar{q}_\beta(y, m, \Gamma)\,\sigma^-\,, && Q^+_{\alpha\beta} = \bar{q}_\alpha(x, n, \Delta)\,q_\beta(y)\,\sigma^+\,,
\end{aligned}
\tag{4.3}
$$

where $\alpha = 0, \ldots, \Delta$, $\beta = 0, \ldots, \Gamma$, $\sigma^+ = (\sigma^-)^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and

$$
q_\alpha(x) = x^\alpha\,, \qquad \bar{q}_\alpha(x, n, \Delta) = \prod_{k=1}^{\Delta-\alpha} (x\partial_x - n + \Delta - k)\,\partial_x^\alpha\,,
$$

with the convention that a product with its lower limit greater than the upper one is defined to be 1. Let $\mathfrak{s}^{\Delta,\Gamma}$ be the abstract Lie superalgebra corresponding to the realization $\mathfrak{s}^{\Delta,\Gamma}_{n,m}$ (it may be shown that the supercommutators (4.2) of the operators (4.3) do not depend explicitly on n and m, [1]). Just as in the single-variable case, [5], the Lie superalgebra $\mathfrak{s}^{\Delta,\Gamma}$ will be infinite-dimensional unless the gaps Δ and Γ are suitably restricted. The corresponding finite-dimensional Lie superalgebras are easily identified.

Lemma 4.1 *The Lie superalgebra $\mathfrak{s}^{\Delta,\Gamma}$ is finite-dimensional if and only if $\Delta + \Gamma \leq 1$. We have:*

$$
\mathfrak{s}^{1,0} \simeq \mathfrak{s}^{0,1} \simeq \mathfrak{spl}(2,1) \oplus \mathfrak{sl}_2\,, \qquad \mathfrak{s}^{0,0} \simeq \bigoplus_{i=1}^{3} \mathfrak{sl}_2^i
$$

Remarks. We have not included in $\mathfrak{s}^{1,0}$ the central term given by K. We regard $\mathfrak{s}^{0,0}$ as a Lie algebra with J and K replaced by $\tilde{J} = \sigma_3$. Finally, recall the isomorphism $\mathfrak{spl}(2,1) \simeq \mathfrak{osp}(2,2)$, [18].

Let $\boldsymbol{\mathfrak{R}}^{n-\Delta,m}_{n,m-\Gamma}$ be the associative subalgebra of $\boldsymbol{\mathfrak{D}}$ spanned by the operators which preserve $\boldsymbol{\mathcal{R}}^{n-\Delta,m}_{n,m-\Gamma}$. It is easy to see that $\mathfrak{s}^{\Delta,\Gamma}_{n,m} \subset \boldsymbol{\mathfrak{R}}^{n-\Delta,m}_{n,m-\Gamma}$, and that the action is irreducible. Thus, the Burnside Theorem guarantees that any matrix differential operator preserving this module may be expressed as a

polynomial in the generators of $\mathfrak{s}^{\Delta,\Gamma}_{n,m}$ plus an operator annihilating the module. The kernel of the homomorphism $\pi^{\Delta,\Gamma}_{n,m} : \boldsymbol{\mathfrak{R}}^{n-\Delta,m}_{n,m-\Gamma} \to \operatorname{End}(\boldsymbol{\mathcal{R}}^{n-\Delta,m}_{n,m-\Gamma})$ is simply the set of operators (4.1) whose component T_{ij} belongs to $\ker \pi_{n_j,m_j}$ as given in Lemma 3.2. We choose a distinguished complement $\tilde{\boldsymbol{\mathfrak{R}}}^{n-\Delta,m}_{n,m-\Gamma}$ to $\ker \pi^{\Delta,\Gamma}_{n,m}$ just as in the scalar case. The following lemma is a direct generalization of the corresponding single-variable result, [5]:

Lemma 4.2 *Let $\bar{\rho}^{\Delta,\Gamma}_{n,m} : \mathfrak{U}(\mathfrak{s}^{\Delta,\Gamma}) \to \boldsymbol{\mathfrak{D}}$ be the homomorphism determined by the representation $\rho^{\Delta,\Gamma}_{n,m} : \mathfrak{s}^{\Delta,\Gamma} \to \mathfrak{s}^{\Delta,\Gamma}_{n,m}$. The monomials*

$$\begin{aligned}\big\{X(S^{\pm})^i(S^0)^s(T^{\pm})^j(T^0)^t\,,\ Q^{\varepsilon}_{\alpha 0}(S^{\varepsilon^*})^i(T^{\pm})^j(T^0)^t\,,\\ Q^{\varepsilon}_{0\beta}(S^{\pm})^i(S^0)^s(T^{\varepsilon})^j\,,\ Q^{\varepsilon}_{\alpha\beta}(S^{\varepsilon^*})^i(T^{\varepsilon})^j\big\}\,,\end{aligned} \tag{4.4}$$

where $X = Q^{\varepsilon}_{00}$, $\mathbb{1}$, $\tilde{J}$, $\varepsilon = -\varepsilon^* = \pm$, $\alpha = 1,\dots,\Delta$, and $\beta = 1,\dots,\Gamma$, form a basis of $\operatorname{im}\bar{\rho}^{\Delta,\Gamma}_{n,m}$.

Note that $\tilde{J}$ may be expressed in terms of $\mathbb{1}$, J, K (or $Q^{\pm}_{00}$ for $\mathfrak{s}^{0,0}$). The number of monomials in (4.4) acting non-trivially in $\boldsymbol{\mathcal{R}}^{n-\Delta,m}_{n,m-\Gamma}$ is precisely $\dim \tilde{\boldsymbol{\mathfrak{R}}}^{n-\Delta,m}_{n,m-\Gamma}$.

Theorem 4.3 *Let $T^{(k)}$ be a k-th order differential operator in $\tilde{\boldsymbol{\mathfrak{R}}}^{n-\Delta,m}_{n,m-\Gamma}$. Then $T^{(k)}$ may be expressed as a linear combination of the monomials in* (4.4) *of differential order at most k.*

4.2 The triangular module $\mathcal{T}_{n_1} \oplus \mathcal{T}_{n_2}$

We shall assume that $n = n_2 \geq n_1$, and let $\Delta = n - n_1$. Let us denote the module $\mathcal{T}_{n-\Delta} \oplus \mathcal{T}_n$ by $\boldsymbol{\mathcal{T}}^{n-\Delta}_n$. Let $\boldsymbol{\mathfrak{T}}^{n-\Delta}_n$ be the associative subalgebra of $\boldsymbol{\mathfrak{D}}$ of differential operators preserving $\boldsymbol{\mathcal{T}}^{n-\Delta}_n$. Consider the $9 + (\Delta+1)(\Delta+2)$ matrix differential operators given by

$$\begin{aligned}&T^i = \operatorname{diag}(J^i_{n-\Delta}, J^i_n)\,, \qquad J = \frac{1}{3}\operatorname{diag}(n+2\Delta, n)\,, \qquad i = 1,\dots,8\,,\\ &Q^-_{\alpha\beta} = q_{\alpha\beta}\,\sigma^-\,, \qquad Q^+_{\alpha\beta} = \bar{q}_{\alpha\beta}(n,\Delta)\,\sigma^+\,, \qquad 0 \leq \alpha+\beta \leq \Delta\,,\end{aligned} \tag{4.5}$$

with J^i_n given in (2.3), and

$$q_{\alpha\beta} = x^{\alpha}y^{\beta}\,, \qquad \bar{q}_{\alpha\beta}(n,\Delta) = \prod_{k=1}^{\Delta-\alpha-\beta}(x\partial_x + y\partial_y - n + \Delta - k)\,\partial_x^{\alpha}\,\partial_y^{\beta}\,.$$

We observe that the even generators in (4.5) span a Lie algebra which is isomorphic to $\mathfrak{gl}_3$. It may be easily verified that both the generators $Q^-_{\alpha\beta}$ and $Q^+_{\alpha\beta}$ span irreducible modules under the adjoint action of the even subalgebra, which are isomorphic to the standard cyclic $\mathfrak{sl}_3$-module of highest weight $(\Delta, 0)$, with J acting as a scalar. Moreover, the anticommutator of $Q^-_{\alpha\beta}$ and $Q^+_{\gamma\delta}$ can be expressed as a polynomial of degree Δ in the even generators and the identity, depending explicitly on n only through a multiple of the identity. Let us denote by $\mathfrak{s}^\Delta_n$ the Lie superalgebra generated by the differential operators (4.5).

Lemma 4.4 *The Lie superalgebra $\mathfrak{s}^\Delta_n$ is finite-dimensional if and only if $\Delta \le 1$. We have:*

$$\mathfrak{s}^1_n \simeq \mathfrak{spl}(3,1)\,, \qquad \mathfrak{s}^0_n \simeq \mathfrak{sl}_3 \oplus \mathfrak{sl}_2\,.$$

Remark. We regard $\mathfrak{s}^0_n$ as a Lie algebra with J replaced by $\tilde{J} = \sigma_3$.

In general, the Lie superalgebra of differential operators preserving a pair of N-dimensional "triangular" modules with a gap $\Delta = 1$ is isomorphic to $\mathfrak{spl}(N+1,1)$. It is easy to see that $\mathfrak{s}^\Delta_n \subset \mathfrak{T}^{n-\Delta}_n$, and that the action on $\boldsymbol{\mathcal{T}}^{n-\Delta}_n$ is irreducible. The kernel of the homomorphism $\pi^\Delta_n : \mathfrak{T}^{n-\Delta}_n \to \mathrm{End}(\mathcal{T}^{n-\Delta}_n)$ is just the set of the operators (4.1) with entries T_{ij} in $\ker \pi_{n_j}$ as given in Lemma 3.2. As usual, we choose as distinguished complement to $\ker \pi^\Delta_n$ the vector space $\tilde{\mathfrak{T}}^{n-\Delta}_n$ spanned by the differential operators in $\mathfrak{T}^{n-\Delta}_n$ with components of the form $T_{ij} = \sum_{s+t\le n_j} f_{st}(x,y)\,\partial^s_x\partial^t_y$.

We first note that the definition of bidegree can be naturally extended to matrix-valued differential operators acting on $\mathbb{C}[x,y]\oplus\mathbb{C}[x,y]$. The bidegrees of the generators (4.5) of $\mathfrak{s}^\Delta_n$ are well-defined. The bidegree of each T^i is given by the bidegree of the corresponding J^i_n, and $\deg J = (0,0)$. For the generators of the odd subspace, we have $\deg Q^\pm_{\alpha\beta} = \mp(\alpha,\beta)$. In order to define the length of a monomial in the generators (4.5) we need to be a little more careful. Let $T^K = (T^1)^{k_1}\cdots(T^8)^{k_8}$. Consider the monomials of the form $T^K_\epsilon = X_\epsilon T^K$, where $\epsilon = 1, 2, \pm$, $X_1 = P_1$, $X_2 = P_2$, $X_\pm = Q^\pm_{\alpha\beta}$, and P_1, P_2 are the canonical projectors, which can of course be expressed in terms of $\mathbb{1}$ and J (or $\tilde{J}$, if $\Delta = 0$). We define the length of a monomial T^K_ϵ as

$$|T^K_\epsilon| = \min_{\deg T^L_\epsilon = \deg T^K_\epsilon} |L|\,,$$

where the minimum is taken over the set of monomials T^L_ϵ with $|L| \le |K|$. Again, a monomial T^K_ϵ is of maximal length if $|T^K_\epsilon| = |K|$. Thus, P_1, P_2 and

the $Q^{\pm}_{\alpha\beta}$'s are all of maximal length (equal to zero). For $\epsilon = \pm$, the bidegree does *not* determine uniquely the monomials of maximal length. The structure

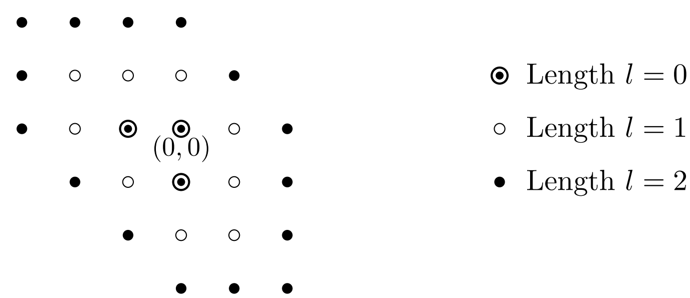

Figure 3: Lattice of bidegrees of monomials T^K_+ of length $l \le 2$ for $\Delta = 1$.

theorem for $\tilde{\mathfrak{T}}^{n-\Delta}_n$ may now be stated as follows:

Theorem 4.5 *Let $T^{(k)}$ be a k-th order differential operator in $\tilde{\mathfrak{T}}^{n-\Delta}_n$. Then $T^{(k)}$ may be expressed as a linear combination of the monomials*

$$\left\{ T^L_\epsilon \, (T^7)^i (T^8)^j \;\middle|\; \epsilon = 1, 2, \pm\,,\ 0 \le i + j + |L| \le t \le k - \delta_{\epsilon +}\Delta \right\}, \tag{4.6}$$

where T^L_ϵ are monomials of maximal length and different bidegrees for each (fixed) value of ϵ.

Remark. If $\Delta > k$, no monomial $T^L_+ \, (T^7)^i (T^8)^j$ may appear in the expansion of $T^{(k)}$.

Proof. The monomials (4.6) are linearly independent. Let N^ϵ_k denote the number of monomials of order at most k and type ϵ in (4.6). We already know that $N^{1,2}_k = \frac{1}{4}(k+1)^2(k+2)^2$, cf. (3.6). Let $\tau(n_0) = \dim \mathcal{T}_{n_0} = \frac{1}{2}(n_0+1)(n_0+2)$. An easy graphical argument, cf. Fig. 3, shows that

$$\begin{aligned} N^{\pm}_k &= \frac{(\Delta+1)(\Delta+2)}{2}\, \tau(k - \delta_{\epsilon +}\Delta) + \sum_{l=1}^{k - \delta_{\epsilon +}\Delta} 3(2l + \Delta)\, \tau(k - \delta_{\epsilon +}\Delta - l) \\ &= \frac{1}{4}(k+1)(k+2)(k \mp \Delta + 1)(k \mp \Delta + 2)\,. \end{aligned} \tag{4.7}$$

Therefore, $N^1_{n-\Delta} + N^2_n + N^+_n + N^-_{n-\Delta} = \dim \tilde{\mathfrak{T}}^{n-\Delta}_n$. The theorem follows from the fact that the highest order derivatives of the monomials (4.6) are all independent. Q.E.D.

The dimension of the subspace $\tilde{\mathfrak{T}}_n^{n-\Delta,(k)}$ of $\tilde{\mathfrak{T}}_n^{n-\Delta}$ of differential operators of order at most k is given by

$$\dim \tilde{\mathfrak{T}}_n^{n-\Delta,(k)} = \frac{1}{2}(k+1)(k+2)(2k^2+6k+4+\Delta^2)\,.$$

It is worth mentioning that in the one-dimensional rank two case this number is independent of Δ, [5].

4.3 The staircase module $\mathcal{S}^r_{p_1,q_1} \oplus \mathcal{S}^r_{p_2,q_2}$

For the sake of simplicity, we shall restrict ourselves to the case $p = p_1+1 = p_2$, $q = q_1 = q_2$. Let us denote the module $\mathcal{S}^r_{p-1,q} \oplus \mathcal{S}^r_{p,q}$ by $\boldsymbol{\mathcal{S}}^{r,1}_{p,q}$. Consider the finite-dimensional Lie superalgebra $\mathfrak{s}^r_p$ spanned by the $10+2r$ matrix differential operators given by:

$$\begin{aligned}
&T^i = \operatorname{diag}(J^i_{p-1}, J^i_p)\,, \qquad J = \frac{1}{2}\operatorname{diag}(p+1,p)\,, \qquad i = 1,\dots,5+r\,,\\
&Q^-_0 = \sigma^-\,, \qquad Q^-_1 = x\,\sigma^-\,, \qquad Q^+_0 = (x\partial_x + ry\partial_y - p)\,\sigma^+\,, \qquad Q^+_1 = \partial_x\,\sigma^+\,,\\
&Q^+_{2+\alpha} = x^\alpha \partial_y\,\sigma^+\,, \qquad \alpha = 0,\dots,r-1\,,
\end{aligned} \tag{4.8}$$

with J^i_p given in (2.4). The abstract structure of the Lie superalgebras $\mathfrak{s}^r_p$ is the semidirect product $\mathfrak{pl}(2,1) \ltimes \mathbb{C}^{r+1,r}$, where $\mathbb{C}^{r+1,r}$ corresponds to the Abelian Lie superalgebra spanned by T^{5+i} and $Q^+_{2+\alpha}$, $i = 0,\dots,r$, $\alpha = 0,\dots,r-1$. It may be readily verified that $\mathfrak{s}^r_p$ is contained in the associative subalgebra $\boldsymbol{\mathfrak{S}}^{r,1}_{p,q}$ of $\boldsymbol{\mathfrak{D}}$ of differential operators preserving $\boldsymbol{\mathcal{S}}^{r,1}_{p,q}$. The kernel of the homomorphism $\pi^{r,1}_{p,q} : \boldsymbol{\mathfrak{S}}^{r,1}_{p,q} \to \operatorname{End}(\boldsymbol{\mathcal{S}}^{r,1}_{p,q})$ is spanned by the differential operators (4.1) with components T_{ij} in $\ker \pi^r_{p_j,q}$ as given in Lemma 3.2. We choose as a distinguished complement to $\ker \pi^{r,1}_{p,q}$ the subspace $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1}_{p,q} \subset \boldsymbol{\mathfrak{S}}^{r,1}_{p,q}$ of differential operators with components of the form:

$$T_{ij} = \sum_{\substack{0\le s+rt\le p_j\\ 0\le t\le q}} f_{st}(x,y)\,\partial^s_x \partial^t_y\,.$$

We note that just as in the scalar case, the module $\boldsymbol{\mathcal{S}}^{r,1}_{p,q}$ is reducible (but not completely reducible) under the action of $\mathfrak{s}^r_p$. In fact, no operator in $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1}_{p,q}$ with positive y-degree may be obtained from a polynomial in the generators of $\mathfrak{s}^r_p$. Let $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1\downarrow}_{p,q}$ denote the subspace of $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1}_{p,q}$ of differential operators with

nonpositive y-degree. The structure theorem for $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1\downarrow}_{p,q}$ may be proved by essentially the same argument used in Theorem 3.10.

Theorem 4.6 *Let $T^{(k)}$ be a k-th order differential operator in $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1\downarrow}_{p,q}$. If $k \leq q$, then $T^{(k)}$ may be represented as a $(k+1)$-th degree polynomial in the generators of $\mathfrak{s}^r_p$. If $k > q$, then $T^{(k)}$ may be expressed as the projection of such a polynomial in $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1}_{p,q}$ along* $\ker \pi^{r,1}_{p,q}$.

Proof. The key of the proof is to obtain a suitable set of monomials in the generators of $\mathfrak{s}^r_p$ such that their projections in $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1}_{p,q}$ along $\ker \pi^{r,1}_{p,q}$ are all independent. We consider monomials of the form

$$T^L_\epsilon (T^3)^i (T^4)^j, \tag{4.9}$$

where $\epsilon = 1, 2\pm$, and T^L_ϵ are monomials of maximal length and different bidegrees for any (fixed) value of ϵ. For $\epsilon = 1, 2$, Lemma 3.9 implies that we can obtain $\dim \tilde{\mathfrak{S}}^{r\downarrow}_{p_\epsilon,q}$ monomials of the form (4.9) with independent projections in $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1}_{p,q}$ along $\ker \pi^{r,1}_{p,q}$. Let $\tilde{\mathfrak{S}}^{r\downarrow -}_{p,q}$ denote the subset of $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1\downarrow}_{p,q}$ of differential operators mapping $P_1(\boldsymbol{\mathcal{S}}^{r,1}_{p,q})$ into $P_2(\boldsymbol{\mathcal{S}}^{r,1}_{p,q})$. The projections in $\tilde{\boldsymbol{\mathfrak{S}}}^{r,1}_{p,q}$ along $\ker \pi^{r,1}_{p,q}$ of the monomials

$$\left\{ Q^-_\epsilon (T^{1+\epsilon})^s (T^{5+\epsilon r})^j (T^3)^n (T^4)^m,\, Q^-_0 (T^5)^t T^{5+i} (T^{5+r})^{j-t-1} (T^3)^n (T^4)^m \right\}, \tag{4.10}$$

with $\epsilon = 0, 1$, $0 \leq j \leq q$, $0 \leq s \leq p-1-jr$, $0 \leq t \leq j-1$, $1 \leq i \leq r - \delta_{t0}$, $0 \leq n$, $0 \leq m \leq q-j$, and $n + rm \leq p - \tilde{s} - jr$ (where $\tilde{s} = s$ for the first group of monomials, and $\tilde{s} = 0$ for the second group), form a basis of $\tilde{\mathfrak{S}}^{r\downarrow -}_{p,q}$. The number of monomials in (4.10) is given by

$$N^{r-}_{pq} = \sum_{j=0}^{q} \sum_{s=0}^{p-1-jr} f(j,s) \sigma\big(r, p-1-s-jr, \min\big(q, [\frac{p-1-s}{r}]\big) - j\big),$$

where $f(j,s) = 2$ if $s \geq 1$ and $f(j,0) = jr + 2$, and $\sigma(r_0, p_0, q_0) = \dim \mathcal{S}^{r_0}_{p_0,q_0}$. It may be verified that

$$N^{r-}_{pq} = \dim \tilde{\mathfrak{S}}^{r\downarrow -}_{p,q} = \frac{(q+1)(q+2)}{2} \Big(p(p+1) - qr(p + \frac{2}{3}) + \frac{qr^2}{12}(3q+1) \Big).$$

We also leave as an exercise to the reader to obtain

$$N_{pq}^{r+} = \dim \tilde{\mathfrak{S}}_{p,q}^{r\downarrow+} = \frac{(q+1)(q+2)}{2}\Big(p(p+1) - qr(p+\frac{1}{3}) + \frac{qr^2}{12}(3q+1)\Big)$$

monomials of the form $T_+^L\,(T^3)^i(T^4)^j$ with independent projections in $\tilde{\boldsymbol{\mathfrak{S}}}_{p,q}^{r,1}$ along $\ker \pi_{p,q}^{r,1}$. The theorem follows from the fact that the highest order derivatives of the monomials (4.9) are all different. Q.E.D.

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